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Noncommutativity and tachyon condensation

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Abstract: We study the fuzzy or noncommutative Dp-branes in terms of infinitely many unstable D0-branes, from which we can construct any Dp-branes. We show that the tachyon condensation of the unstable D0-branes induces the noncommutativity. In the infinite tachyon condensation limit, most of the unstable D0-branes disappear and remaining D0-branes are actually the BPS D0-branes with the correct noncommutative coordinates. For the fuzzy S^2 case, we explicitly show only the D0-branes corresponding to the lowest Landau level survive in the limit. We also show that a boundary state for a Dp-brane satisfying the Dirichlet boundary condition on a curved submanifold embedded in the flat space is not localized on the submanifold. This implies that the Dp-brane on it is ambiguous at the string scale and solves the problem for a spherical D2-brane with a unit flux on the world volume which should be equivalent to one D0-brane. We also discuss the diffeomorphism in the D0-brane picture.

Keywords: D-branes, Non-Commutative Geometry, M(atrix) Theories, Tachyon Condensation.

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Contents

1. Introduction

Noncommutative generalization of the geometry is an interesting subject partly because it is expected to be related to a quantization of the general relativity which is based on riemannian geometry. It has been shown that some noncommutative geometry, for examples, noncommutative torus, noncommutative plane and fuzzy sphere play important roles in D-brane physics in string theory [1−4]. It should be emphasized that in these examples the noncommutative Dp-brane can be equivalently described as BPS D0-branes with matrix valued coordinate Φ^μ which do not commute each other and this may be the origin of the noncommutativity.

On the other hand, a noncommutative generalization of the Riemann geometry was given by Connes [5]. In this noncommutative geometry, the spectral triples which include (a generalization of) the Dirac operator play a central role. In string theory, remarkably, the spectral triples of the noncommutative geometry á la Connes can be identified as configurations of the unstable D0-branes like D0$-\overline{\mathrm{D0}}$ branes pairs (or D(-1) branes if we consider euclidean space-time). [6 − 9]. Actually, we can construct any D-branes from infinitely many unstable D0-branes.[1] Therefore this unstable D0-brane picture gives unified view for the all D-branes, including pure D-branes, the D-branes on the fuzzy sphere and the flat noncommutative D-branes. Therefore, the system of the unstable D0-branes gives a unified picture of the all D-branes in the theory and could be a starting point to consider a nonperturbative definition of a string theory [6]-[8, 10] which can be regarded as a generalization of the matrix model [11, 12].[2] Then, it is natural to ask how this unstable D0-brane picture incorporates the noncommutativity of the fuzzy D-branes, etc, which are represented as the mutually noncommutative matrix coordinates in the BPS D0-brane picture. This question is very interesting since the matrix coordinates of the unstable D0-branes corresponding to the Dp-brane with the flux are mutually commutative [9, 8] and there does not seem noncommutativity.

In this paper, we show that the tachyon condensation of the unstable D0-branes induces the noncommutativity. We will see that in the infinite tachyon condensation limit, in which the corresponding Dp-brane boundary state becomes the usual form, most of unstable D0-branes disappear by the tachyon condensation. Indeed, the remaining D0-branes consist the BPS D0-branes with the correct noncommutative coordinates as we will see in the fuzzy S^2 case explicitly. The noncommutativity appears because of the disappearance of the D0-branes by the condensation of the tachyon, which does not commute with the matrix coordinates.

The fuzzy S^2 brane corresponding to the D2-brane on S^2 with a unit flux should be built from one BPS D0-brane, however, a BPS D0-brane can not be fuzzy. Thus we might expect that there is no D2-brane picture for this “fuzzy” brane. This might be a problem for the unified unstable D0-brane picture, however, we also show that the boundary state for the Dp-brane on the curved manifold which satisfies the Dirichlet boundary condition on the manifold is not localized on the manifold. Thus the D2-brane on S^2 with unit flux is equivalent to a BPS D0-brane even though it is at the origin. Here the D2-brane on S^2 means that it satisfies the Dirichlet boundary condition on the S^2, but in reality it is at the origin. Then we have another problem which is constructing the Dp-brane localized on the curved manifold. Our result indicates that the boundary state for the Dp-brane localized on the curved manifold should have a nonzero boundary term which corresponds to the tachyon of the unstable D0-branes. This implies that the Dp-brane on the curved manifold is ambiguous at the string scale.

The organization of the paper is as follows. In section 2, we review how to construct Dp-branes from infinitely many unstable D0-branes. In section 3, we show for some cases the

[1] Recently, D-branes are realized as solitons in the gauge theory with tachyon fields defined on higher dimensional unstable D-brane systems [46]. More recently, it was shown that D-branes can also be constructed as bound states of a lower dimensional unstable D-brane system [9, 6]. For early works on tachyon condensation in string theory, see [47].

[2] The construction of the Dp-brane from the unstable D0-branes was shown only for the flat space-time background although we expect generalizations to other background is possible. In this paper, we will consider the flat space-time background only.

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unstable D0-branes are reduced to fewer number of D0-branes by the tachyon condensation. We consider the fuzzy S^2 case in which finite number of the D0-branes consist the D2-branes. We also consider the flat noncommutative D($2p$)-branes. In section 4, we show the tachyon condensation of the unstable D0-branes induces the fuzziness to the world volume of the Dp-brane. We also discuss the diffeomorphism and Seiberg-Witten map in terms of D0-branes. Finally, we conclude our study in this paper and discuss some future problems in section 5. In appendix A, we briefly review the boundary states. In appendix B, we summarize the normalization of the boundary state. In appendix C, we discuss the validity of the assumption made in section 3 and study the Harmonic oscillator case.

2. Dp-brane from infinitely many unstable D0-branes

In this section, we will explain how to construct Dp-branes from infinitely many unstable D0-branes.[3]

First we will extend the construction of the Dp-branes from infinitely many unstable D0-branes in superstring theory [9, 8] to that in bosonic string theory. The tachyons and other fields on N D0-branes in bosonic string become hermitean $N \times N$ matrices. Taking a large-N limit, they may become operators acting on some Hilbert space and each base element of an orthonormal basis of it correspond to a D0-brane. Let us consider the usual Hilbert space of quantum mechanics, on which the operators $\widehat{p}$ and $\widehat{x}$ act. Here they satisfy the usual commutation relation $[\widehat{x}, \widehat{p}] = i$.[4] Note that this Hilbert space should be distinguished from (the zero mode sector of) the closed string Hilbert space used in the boundary state. Thus we will use two different types of operators which act different Hilbert spaces.

Let us consider a following configuration of the tachyon T and the massless scalars Φ^μ, which represent a position of the D0-branes:

$$T = u^2 \widehat{H}_0, \quad \Phi^\alpha = \widehat{x}^\alpha\,, \qquad (\alpha = 1, \dots, p)\,, \tag{2.1}$$

where $\widehat{H}_0 = \frac{1}{2}\sum_{\alpha=1}^{p}(\widehat{p}_\alpha)^2$ and other fields were set to be zero. Taking $u \to \infty$ limit, the configuration (2.1) will be localized near $\widehat{p}_\alpha = 0$ because the tachyon potential vanishes as $T \to \infty$ and the unstable D0-branes with $\widehat{p}^2 > 0$ disappear. Actually the tachyon potential of the BSFT action [13, 14] is like $V(T) = \mathrm{Tr}(e^{-T}(1+T))$ which goes to zero as $T \to \infty$.

This is a bosonic string analogue of the configuration of the unstable D0-branes in superstring theory which represents the Dp-brane [9]. It can be obtained from the usual kink configuration $T = ux$ in a non BPS D1-brane representing a BPS D0-brane [15−17] by replacing x with $\widehat{p}$ and taking $\Phi^\alpha = \widehat{x}^\alpha$. Indeed, in bosonic case, (2.1) is also obtained from the unstable kink configuration $T = u^2 x^2$ in D25-brane [13, 14], which represents D24-brane, by the same procedure. Therefore, in the same reason as in superstring case [9], (2.1) will be a Dp-brane solution of the equations of motion for the BSFT action [13, 14] in the

[3] In this paper, we set $\alpha' = 1$ or $\alpha' = 2$ for bosonic string string or superstring theory, respectively, unless we recover explicit α' dependence.

[4] This system does not have normalized basis, however, we can consider the harmonic oscillator system in appendix C as a regularization of the system.

$u \to \infty$ limit. In this paper, instead of using BSFT action, we will use the boundary state to show this configuration indeed represents the Dp-brane as in the superstring case [6−8]. See appendix A for a brief summary of the boundary state used in this paper.

The boundary state for the large-N D0-branes with (2.1) is given by

$$e^{-S_b}\,|x\!=\!0\rangle = \mathrm{Tr}\,\mathrm{P}\,e^{-i\int d\sigma \hat{H}(\hat{x},\hat{p})}\,|x\!=\!0\rangle\ , \tag{2.2}$$

where

$$i\widehat{H}(\widehat{x},\widehat{p}) = \frac{u^2}{2}\sum_{\alpha=1}^{p}(\widehat{p}_\alpha)^2 + i\widehat{x}^\alpha P_\alpha(\sigma)\,, \tag{2.3}$$

$|x\!=\!0\rangle$ is the boundary state for a D0-brane at the origin and e^{-S_b} is the boundary interaction (A.16). Since this is a quantum mechanical partition function with hamiltonian $\widehat{H}$ which includes the time-dependent perturbation $i\widehat{x}P(\sigma)$, we can rewrite it in terms of the path-integral formulation as in [8, 19]. The lagrangian corresponding to $\widehat{H}$ is

$$iL(x,\dot{x}) = -\frac{1}{2u^2}(\dot{x}^\alpha(\sigma))^2 - ix^\alpha(\sigma)P_\alpha(\sigma)\,, \tag{2.4}$$

and

$$e^{-S_b}\,|\,x=0\,\rangle = \int [dx^\alpha(\sigma)]e^{i\int d\sigma L(x,\dot{x})}\,|\,x=0\,\rangle = \int [dx^\alpha(\sigma)]e^{-\int d\sigma\frac{1}{2u^2}(\dot{x}^\alpha)^2}\,|\,x^\alpha(\sigma)\,\rangle\ . \tag{2.5}$$

Here, we set $p = 25$ for simplicity, however, a generalization to a Dp-brane is obvious. Thus, in the $u \to \infty$ limit, we have

$$\lim_{u\to\infty} e^{-S_b}\,|\,x=0\,\rangle = \int [dx^\alpha(\sigma)]\,|\,x^\alpha(\sigma)\,\rangle\ . \tag{2.6}$$

Because the boundary state for a D25-brane can be written as $|\,D25\,\rangle = \int [dx^\alpha(\sigma)]\,|\,x^\alpha(\sigma)\,\rangle$, we have exactly proved that the Dp-brane is equivalent to the infinitely many D0-branes with (2.1).

From the apparent similarity between this and the one in [8], we can easily incorporate the tachyon t, gauge field A_α and massless scalars ϕ^i ($i = p+1,\dots,25$) on the Dp-brane. Actually, taking

$$T = u^2\widehat{H}_0\,,\qquad \Phi^\alpha = \widehat{x}^\alpha\,,\qquad \Phi^i = \phi^i(\widehat{x})\,, \tag{2.7}$$

with

$$\widehat{H}_0 = \frac{1}{2}\sum_{\alpha=1}^{p}(\widehat{p}_\alpha - iA_\alpha(\widehat{x}))^2 + t(\widehat{x})\,, \tag{2.8}$$

we have

$$i\widehat{H} = \frac{u^2}{2}(\widehat{p}_\alpha - iA_\alpha(\widehat{x}))^2 + t(\widehat{x}) + i\phi^i(\widehat{x})P_i + i\widehat{x}^\alpha P_\alpha\,, \tag{2.9}$$

and a corresponding lagrangian

$$iL(x,\dot{x}) = -\frac{1}{2u^2}\dot{x}^2 - A_\alpha\dot{x}^\alpha - t(x) - i\phi^i(x)P_i - ix^\alpha P_\alpha\,. \tag{2.10}$$

Thus, we obtain the correct boundary state of the Dp-brane with A_α, t, ϕ^i:

$$e^{-S_b}\,|\,x=0\,\rangle \;=\; \int [dx(\sigma)] e^{\int d\sigma\left(-\frac{1}{2u^2}\dot{x}^2 - A_\alpha \dot{x}^\alpha - t(x)\right)}\,|\,x^\alpha(\sigma), \phi^i(x^\alpha(\sigma))\,\rangle \tag{2.11}$$

$$\xrightarrow{u\to\infty} \int [dx(\sigma)] e^{\int d\sigma(-A_\alpha \dot{x}^\alpha - t(x))}\,|\,x^\alpha(\sigma), \phi^i(x^\alpha(\sigma))\,\rangle\,. \tag{2.12}$$

Therefore we have two different descriptions of the same physical system: one using the infinitely many D0-branes and the one using the Dp-brane. In other words, we can construct any Dp-brane from the D0-branes as in superstring case [9]. Moreover, we can easily extend this construction of a Dp-brane to several Dp-branes. We can also generalize this construction to curved Dp-branes by considering a hamiltonian of a particle on a curved space with gauge coupling and a potential as in [8]. The fluctuations of the metric of the world volume can be included as in [8]. We will discuss this world volume metric and the diffeomorphism later.

Finally we will briefly summarize the result in [6, 8] for the superstring. For simplicity, we consider N D0$-\overline{\text{D0}}$ pairs in type-IIA although other cases, including non BPS D0-branes, were discussed in [6, 8]. We always consider N pairs of D0$-\overline{\text{D0}}$ such that the Φ^μ of a D0 and a $\overline{\text{D0}}$ in any pair of D0$-\overline{\text{D0}}$ are same, i.e. the D0 and the $\overline{\text{D0}}$ in the pair are at a same position. The gauge symmetry for the N pairs of D0$-\overline{\text{D0}}$ is $\mathrm{U}(N)\times\mathrm{U}(N)$. The action of the diagonal $\mathrm{U}(N)$ part of it keeps the above condition for the position of D0$-\overline{\text{D0}}$ and in what follows $\mathrm{U}(N)$ symmetry means this diagonal $\mathrm{U}(N)$.

Let us consider the flat BPS D($2p$)-brane [9] in terms of N D0$-\overline{\text{D0}}$ pairs in the $N\to\infty$ limit. For this, the tachyon operator is given by $T = uD$ where $D = \sum_{\alpha=1}^{2p}\gamma^\alpha(\widehat{p}_\alpha - iA_\alpha(\widehat{x})))$, which is the Dirac operator, and $\Phi^\alpha = \widehat{x}^\alpha$ ($\alpha = 1,\ldots,2p$), $\Phi^i = 0$ ($i = 2p+1,\ldots,9$). Then the boundary interaction (A.20) is

$$e^{-S_b} = \mathrm{Tr}\,\mathrm{P}\exp\int d\sigma\left(-u^2(\widehat{p}_\alpha - iA_\alpha(\widehat{x}))^2 + \frac{u^2}{4}F_{\alpha\beta}[\gamma^\alpha,\gamma^\beta] - i\widehat{x}^\alpha P_\alpha(\sigma) - iu\Pi_\alpha(\sigma)\gamma^\alpha\right), \tag{2.13}$$

which can be rewritten as

$$e^{-S_b} = \int [dx^\alpha][d\psi^\alpha]\,\mathrm{P}\exp\int d\sigma\left(-\frac{\dot{x}_\alpha^2 + \psi^\alpha\dot{\psi}^\alpha}{4u^2} - A_\alpha\dot{x}^\alpha + \frac{1}{2}F_{\alpha\beta}\psi^\alpha\psi^\beta - ix^\alpha P_\alpha - i\Pi_\alpha\psi^\alpha\right) \tag{2.14}$$

using the path-integral. Thus, in the $u\to\infty$ limit, we have the correct Dp-brane boundary state from the D0$-\overline{\text{D0}}$ boundary state:

$$e^{-S_b}\,|\,x=0\,\rangle\,|\,\psi=0\,\rangle \to \int [dx^\alpha][d\psi^\alpha]\mathrm{P}e^{\int d\sigma\left(-A_\alpha\dot{x}^\alpha + \frac{1}{2}F_{\alpha\beta}\psi^\alpha\psi^\beta\right)}\,|\,x^\alpha, x^i = 0\,\rangle\,|\,\psi^\alpha, \psi^i = 0\,\rangle\,. \tag{2.15}$$

If we want to consider Dp-brane on a curved submanifold in the flat space, we should have the Dirac operator on it. Using the spin connection

$$\omega_{AB,\gamma} = e_A^\alpha e_B^\beta\left(\Gamma^\delta_{\gamma\alpha}g_{\delta\beta} - \delta_{CD}e_\beta^D\partial_\gamma e_\alpha^C\right), \tag{2.16}$$

where e_α^A is a vielbein satisfying $e_\alpha^A \delta_{AB} e_\beta^B = g_{\alpha\beta}$, the Dirac operator is defined by

$$D \equiv -i\gamma^A e_A^\alpha \nabla_\alpha \,, \tag{2.17}$$

where $\{\gamma^A, \gamma^B\} = 2\delta^{AB}$ and

$$\nabla_\alpha = \frac{\partial}{\partial x^\alpha} - \frac{1}{8}\omega_{AB,\alpha}[\gamma^A, \gamma^B] - iA_\alpha \,. \tag{2.18}$$

Then, we take $T = uD$ and $\Phi^\mu = f^\mu(x^\alpha)$, where f^μ is the embedding of the curved submanifold to the flat space. Inserting these into the formula of the boundary interaction (A.20), we have

$$e^{-S_b} = \mathrm{Tr}\,\mathrm{P}\exp\left(-i\int d\sigma \widehat{H}\right) \tag{2.19}$$

with the total hamiltonian

$$i\widehat{H} = u^2 \widehat{H}_0 + +if^\mu(x)P_\mu(\sigma) + iu\Pi_\mu(\sigma)\frac{\partial f^\mu(x)}{\partial x^\alpha}\gamma^\alpha, \tag{2.20}$$

where

$$\widehat{H}_0 = D^2 \,. \tag{2.21}$$

We can rewrite it in the path-integral formalism:

$$e^{-S_b} = \int [dx^\alpha][d\psi^\alpha]\mathrm{P}e^{i\int d\sigma L}\,, \tag{2.22}$$

where

$$iL = -\frac{1}{4u^2}g_{\alpha\beta}(x)(\dot{x}^\alpha\dot{x}^\beta + \psi^\alpha \nabla_\sigma \psi^\beta) - A_\alpha\dot{x}^\alpha + \frac{1}{2}F_{\alpha\beta}\psi^\alpha\psi^\beta - if^\mu(x)P_\mu - i\Pi_\mu\frac{\partial f^\mu(x)}{\partial x^\alpha}\psi^\alpha\,, \tag{2.23}$$

and $\nabla_\sigma\psi^\alpha = \dot{\psi}^\alpha + \dot{x}^\beta\Gamma^\alpha_{\beta\gamma}\psi^\gamma$. Thus in the $u \to \infty$ limit, we obtain a boundary state for the Dp-brane wrapping the submanifold:

$$\int [dx^\alpha][d\psi^\alpha]\mathrm{P}e^{-\int d\sigma\left(A_\alpha\dot{x}^\alpha - \frac{1}{2}F_{\alpha\beta}\psi^\alpha\psi^\beta\right)}\,|\,f^\mu(x)\,\rangle\,|\,\partial_\alpha f^\mu\,\psi^\alpha\,\rangle\;. \tag{2.24}$$

Here the path-integral measure $[dx^\alpha]$ may depend on the vielbein e_α^A. The measure may become usual one in a coordinate $x'^\mu(x)$ such that $g'_{\alpha\beta}(x') \sim \delta_{\alpha\beta}$. Thus we expect that the boundary state (2.24) will be the Dp-brane boundary state if we choose the vielbein such that the metric $g_{\alpha\beta} = e_\alpha^A e_\beta^A$ is the induced metric $\frac{\partial f^\mu}{\partial x^\alpha}\frac{\partial f^\mu}{\partial x^\beta}$.

The normalization of the boundary states has not been considered. It will be discussed in appendix B.

It should be emphasized that the boundary states we consider in this paper are not BRST invariant in general. Thus they do not represent solutions of the equations of motion nor boundary conformal field theory in general and not called boundary states in the strict sense of the word. However, these boundary states which are naively extended to off-shell field are meaningful, at least, by considering the BSFT action for it which is obtained as $S_{BSFT} = \frac{2\pi}{g_s}\langle\,0\,|\,Dp\rangle$ and considering the couplings to the closed string fields, such as the energy-momentum tensor. In order to get the solutions, we should restrict the boundary integrations. We will see some examples we consider in this paper are on-shell and the boundary states are BRST invariant although we do not consider the restriction in general cases.

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3. Disappearance of D0-branes and fuzzy D-branes

In this section, we will consider a hamiltonian $\widehat{H}_0$ which has a gap above ground states.[5] We take an orthonormalized eigen state $|\,n\,\rangle$ of $\widehat{H}_0$ with the eigen value E_n, i.e. $\widehat{H}_0\,|\,n\,\rangle = E_n\,|\,n\,\rangle$ and $\langle\, n\,|\,n'\rangle = \delta_{n,n'}$. The eigen states span the whole Hilbert space and each eigen state represents an unstable D0-brane. Note that we can choose any base of the Hilbert space. The orthonormalized bases are related each other by a U(N) gauge transformation of the unstable N D0-branes. Especially, the path-integral formalism is closely related to a position basis $\{|\,x^\mu\,\rangle\}$. At $u=0$ we can take $\{|\,x^\mu\,\rangle\}$ basis and this system is a collection of N D0-branes which have a definite position.

In the $u\to\infty$ limit, we expect that only the D0-branes which correspond to the ground states of $\widehat{H}_0$ contribute in the boundary interaction

$$e^{-S_b} = \mathrm{Tr}\,\mathrm{P}\,e^{-i\int d\sigma \hat{H}(\hat{x},\hat{p})} \tag{3.1}$$

because other D0-branes have infinitely large tachyon values and disappear by the tachyon condensation.[6] Denoting the degeneracy of the ground state as N_0 and the ground states as $|\,a\,\rangle$, $(a=1,\ldots,N_0)$, this means we will have

$$e^{-S_b} = \mathop{\mathrm{Tr}}_{N_0\times N_0} \mathrm{P}e^{-i\int d\sigma \tilde{\Phi}^\mu P_\mu(\sigma)} \tag{3.2}$$

in bosonic string or

$$e^{-S_b} = \mathop{\mathrm{Tr}}_{N_0\times N_0} \mathrm{P}e^{-i\int d\sigma\left(\tilde{\Phi}^\mu P_\mu(\sigma)+\frac{1}{2}[\tilde{\Phi}^\mu,\tilde{\Phi}^\nu]\Pi_\mu(\sigma)\Pi_\nu(\sigma)\right)} \tag{3.3}$$

in superstring, where an $N_0\times N_0$ matrix $\tilde{\Phi}^\mu$ is given by

$$(\tilde{\Phi}^\mu)_{ab} = \langle\, a\,|\,\Phi^\mu(x^\alpha)\,|\,b\,\rangle\;, \tag{3.4}$$

which represents the N_0 D0-branes with matrix coordinates $\tilde{\Phi}^\mu$.[7]

In this paper, we assume this reduction is indeed true. (In appendix C we will discuss the validity of this assumption for some cases.) Then, the the original N unstable D0-branes with the boundary interaction is equivalent to the N_0 D0-branes with $(\tilde{\Phi}^\mu)_{ab} = \langle\, a\,|\,\Phi^\mu\,|\,b\,\rangle$ $(a,b=1,\ldots,N_0)$ in this limit. Note that for the D0$-\overline{\mathrm{D0}}$ case we likely have only N_0 $D0$ or only N_0 $\overline{\mathrm{D0}}$ after the tachyon condensation and then there is no tachyon on N_0 D0-branes.

As a simplest example of this phenomenon, let us consider two D0-branes in bosonic string theory and take

$$T = u^2(1_{2\times 2}+\sigma_1)\,, \qquad \Phi^1 = a\sigma_3 \tag{3.5}$$

[5]The spectrum of the free hamiltonian is gapless. In this case, we can deform the hamiltonian such that the hamiltonian has a gap and then consider the free hamiltonian as a limit of the deformed one. Indeed, we will consider a harmonic oscillator hamiltonian, which can be regarded as a IR regularization of the free hamiltonian in the appendix C.

[6]For the bosonic case we have chosen the constant part of t such that the eigen value of T of ground states is zero or positive. Note that t possibly depends on u.

[7]In (3.2) we have set $\phi^i(x)=0$. If we include this, the exponent of the left hand side of (3.2) will become $\tilde{\Phi}^\mu P_\mu(\sigma) + \tilde{\phi}^i P_i(\sigma)$ where $(\tilde{\phi}^i)_{ab} = \langle\, a\,|\,\phi^i(\widehat{x}^\alpha)\,|\,b\,\rangle$.

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(or two non BPS D-branes with $T = \frac{u}{\sqrt{2}}(1_{2\times 2} + \sigma_1)$ and $\Phi^1 = a\sigma_3$ in the type-IIB superstring). At $u = 0$, two D0-branes are at $x^1 = a$ and $x^1 = -a$. After the unitary transformation $U = \frac{1}{\sqrt{2}}(1 + i\sigma_2)$, we have a diagonal tachyon $T = u^2\,\mathrm{diag}(2, 0)$ and $\Phi^1 = a\,\sigma_1$. For $0 < u^2 < \infty$, the system becomes fuzzy D-branes extending around $-a < x^1 < a$. Then at $u = \infty$, only a D0-brane located at $x^1 = 0$ remain since the D0-brane corresponding to the nonzero eigen state of T disappears by the tachyon condensation and $\langle\, 0 \,|\, \Phi^1 \,|\, 0 \,\rangle = 0$ where $\langle\, 0 \,| = (0, 1)$ is the ground state. We can easily see that only the fluctuations proportional to $|\, 0 \,\rangle \langle\, 0 \,|$ correctly survive at $u = \infty$. This example clearly shows that the D-brane becomes fuzzy even with commutative Φ^μ if the tachyon is turned on. Note that a can be taken very large so that they are separated far away each other at $u = 0$. We also note that this is the solution of the equqtions of motion of D0-branes only for $u = 0$ or $u = \infty$. We will discuss this noncommutativity deeper in section 4.

It is more interesting to start from infinitely many unstable D0-branes. We have seen that If $\widehat{H}_0$ is a hamiltonian for a particle on a manifold, there is another description in terms of a Dp-brane. Therefore we conclude that in the $u \to \infty$ limit the N_0 D0-branes with $(\tilde{\Phi}^\mu)_{ab} = \langle\, a \,|\, \Phi^\mu \,|\, b \,\rangle$ $(a, b = 1, \ldots, N_0)$ are equivalent to the Dp-brane. In particular, the hamiltonian corresponding to the Dp-brane with flux on its world volume typically has a gap between the ground states, namely, lowest Landau level, and excited states. For examples, the flat D($2p$)-brane with flux is the noncommutative D($2p$)-brane which has a description in terms of infinitely many D0-branes with noncommutative coordinates and the D2-brane on S^2 with n flux is the fuzzy D2-brane.

The relation between the Dp-brane with flux and the D0-branes were studied recently in [20]. The boundary states used in [20] for superstring case look very similar to those in [6, 8]. In fact, we can see that they are same as the boundary states in [6, 8] although the u dependent term in $\widehat{H}$ was regarded as an auxiliary regularization term in [20], in which the unstable D0-branes with tachyons were not considered. Thus, we can easily translate the results in [20] into our unstable D0-branes picture with some changes of notations. Especially, the D2-brane on S^2 with n flux in bosonic string theory is an interesting example since the $N_0 = |n| + 1$ is finite.

In the following, we will consider a D2-brane on S^2 with n flux in the flat space-time in terms of infinitely many D0$-\overline{\text{D0}}$ branes in type-IIA superstring theory as an example of the equivalence. This provides an interesting example for considering the fuzziness induced by the tachyon condensation since the D2-brane with a unit flux should be equivalent to one BPS D0-brane which obviously is not on S^2, but at the origin.

3.1 Fuzzy S^2 brane from tachyon condensation

First, let us construct the Dirac operator of charged spin $\frac{1}{2}$ particle in a constant magnetic field on S^2 (See, for example, [21, 22]). Taking the coordinate (θ, ϕ) as usual, the dreibein is $e^i_\mu = \mathrm{diag}(1, \sin\theta)$ and the gauge field in the singular gauge is given by $A_\phi = \frac{n}{2}\cos\theta$ where n should be an integer. Thus the Dirac operator (2.17) in this case is

$$D = -i\sigma_x \left(\partial_\theta + \frac{1}{2}\frac{\cos\theta}{\sin\theta} \right) - i\sigma_y \frac{1}{\sin\theta} \left(\partial_\phi - i\frac{n}{2}\cos\theta \right) \tag{3.6}$$

where σ_x, σ_y are the Pauli matrices. We can compute the D^2, which can be regarded as the hamiltonian of the supersymmetric quantum mechanics, as

$$\begin{aligned} D^2 &= -\frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta - \frac{1}{\sin^2\theta}\left(\partial_\phi - i\frac{n}{2}\cos\theta\right)^2 + i\sigma_z\frac{\cos\theta}{\sin^2\theta}\left(\partial_\phi - i\frac{n}{2}\cos\theta\right) + \\ &+ \frac{1}{4}\left(1+\frac{1}{\sin^2\theta}\right) + \sigma_z\frac{n}{2}\,. \end{aligned} \tag{3.7}$$

This system has the three-dimensional rotation symmetry which allow us to define the angular-momentum operators;

$$L_+ = e^{i\phi}\left(\frac{\partial}{\partial_\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial_\phi} + \frac{1}{2\sin\theta}(\sigma_z+n)\right), \tag{3.8}$$

$$L_- = e^{-i\phi}\left(-\frac{\partial}{\partial_\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial_\phi} + \frac{1}{2\sin\theta}(\sigma_z+n)\right), \qquad L_z = -i\frac{\partial}{\partial_\phi}, \tag{3.9}$$

which satisfy $[L_+, L-] = 2L_z, [L_z, L_\pm] = \pm L_\pm$ and $0 = [D, L_z] = [D, L_\pm]$. As in [23], we can rewrite the hamiltonian as

$$D^2 = L^2 + \frac{1}{4} - \frac{n^2}{4}\,. \tag{3.10}$$

By diagonalizing L^2 and denoting $L^2 = j(j+1)$ as usual, we obtain

$$D^2 = \left(j+\frac{1}{2}\right)^2 - \frac{n^2}{4}\,, \tag{3.11}$$

which means $j + \frac{1}{2} \geq |\frac{n}{2}|$ because D is a hermitian operator. Therefore, the ground states are expected to have $j = |\frac{n}{2}| - \frac{1}{2}$ and the degeneracy of the ground state is $2j+1 = |n|$.

It is easy to find the zero-mode of D. Since the $D\sigma_x$ is a diagonal matrix, $D\psi(\theta,\phi) = 0$ is equivalent to

$$\left(\partial_\theta + \frac{1}{2}\frac{\cos\theta}{\sin\theta} \pm \frac{m - \frac{n}{2}\cos\theta}{\sin\theta}\right)\psi_\pm(\theta) = 0\,, \tag{3.12}$$

where $\psi(\theta,\phi) = {}^t(\psi_-(\theta)e^{im\phi}, \psi_+(\theta)e^{im\phi})$. Thus we obtain

$$\psi_\pm(\theta) = \text{const.} \qquad (1-\cos\theta)^{\frac{1}{4}(-1\mp 2m\pm n)}(1+\cos\theta)^{\frac{1}{4}(-1\pm 2m\pm n)}\,, \tag{3.13}$$

which is nonsingular if and only if $-1 \mp 2m \pm n \geq 0$ and $-1 \pm 2m \pm n \geq 0$.[8] This implies $\pm n \geq 1$ and then we should set $\psi_- = 0$ or $\psi_+ = 0$ for $n > 0$ or $n < 0$, respectively. Note that there is no zero mode of the Dirac operator for $n = 0$. This is, of course, consistent with the Lichnerowicz formula. For given n, we can take

$$|n| - 1 \geq 2m \geq -(|n|-1)\,, \tag{3.14}$$

[8]Note that the wave function $\psi(\theta,\phi)$ is singular if $-1\mp 2m\pm n = 0$ or $-1\pm 2m\pm n = 0$ at $\theta = 0$ or $\theta = \pi$, respectively. However, these singularities can be removed by regular gauge transformations, therefore these are harmless.

and $2m$ should be even or odd integer if $|n|-1$ is even or odd integer, respectively, since we are considering the spin $\frac{1}{2}$ particle. Therefore we correctly obtain $|n|$ Dirac zero modes, which we will denote $|\,m\,\rangle$, $(m=-(|n|-1)/2, -(|n|-3)/2, \cdots, (|n|-3)/2, (|n|-1)/2)$.

Let us consider the D0$-\overline{\text{D0}}$ pairs in type-IIA string theory and take $T=\frac{u}{R}D$ and $\Phi^1 = R\sin\theta\cos\phi$, $\Phi^2 = R\sin\theta\cos\phi$, $\Phi^3 = R\cos\theta$. In the limit $u\to\infty$, only the Dirac zero modes $|\,m\rangle$ survive and the D0$-\overline{\text{D0}}$ pairs corresponding to massive modes of the Dirac operator disappear by the tachyon condensation. Therefore we have $|n|$ D0-branes or $|n|$ $\overline{\text{D0}}$-branes for $n>0$ or $n<0$, respectively. (Here we think $-\sigma_3$ as the bilinear of the boundary fermions $[\eta,\bar\eta]$ in [17].) The positions of these D0-branes, $|n|\times|n|$ matrices $(\tilde\Phi^\alpha)_{mm'}$, are given by

$$
\begin{aligned}
(\tilde\Phi^1)_{mm'} &= R\,\langle\, m\,|\sin\theta\cos\phi\,|\, m'\,\rangle\ , \\
(\tilde\Phi^2)_{mm'} &= R\,\langle\, m\,|\sin\theta\sin\phi\,|\, m'\,\rangle\ , \\
(\tilde\Phi^3)_{mm'} &= R\,\langle\, m\,|\cos\theta\,|\, m'\,\rangle\ .
\end{aligned}
\tag{3.15}
$$

Applying the Wigner-Eckart theorem, we know that $\tilde\Phi^\alpha_{mm'}$ is proportional to the generators of the $|n|$ dimensional representation of SU(2), $(J^\alpha)_{mm'}$. By comparing $\langle\, m_{\max}\,|\cos\theta\,|\, m_{\max}\,\rangle$ and $\langle\, m_{\max}\,|\, L_z\,|\, m_{\max}\rangle$, we obtain

$$(\tilde\Phi^\alpha)_{mm'} = \mathrm{sign}(n)\frac{2R}{|n|+1}(J^\alpha)_{mm'}\,. \tag{3.16}$$

Note that

$$((\tilde\Phi^1)^2+(\tilde\Phi^2)^2+(\tilde\Phi^3)^2)_{mm'} = R^2\frac{|n|-1}{|n|+1}\delta_{mm'} \tag{3.17}$$

and $[\tilde\Phi^1,\tilde\Phi^2] = \mathrm{sign}(n)\frac{4iR}{|n|+1}\tilde\Phi^3$. It is interesting to see the effective radius of the S^2 defined from (3.17), $R_{\text{eff}}\equiv R\sqrt{\frac{|n|-1}{|n|+1}}$, is always smaller than R.

On the other hand, according to the previous discussion, we can use the path-integral formalism to represent this D0$-\overline{\text{D0}}$ system and obtain a BPS D2-brane on the two-sphere of radius R with the uniform n flux. Because this D2-brane description and the D0-brane description of the same system should be equivalent, we conclude that the BPS D2-brane on the two-sphere with the uniform n flux is equivalent to the n BPS D0-branes with (3.16) [24] exactly. We note that this equivalence holds for any order in α', i.e. beyond the approximation used in the DBI action, and the number of BPS D0-branes is correct even for finite n while we have $|n|+1$ ground states for n flux in the bosonic string [20]. Of course, this is a consequence of the D0-brane charge conservation and is related to the Atiyah-Singer index theory as shown in [8].

For $n=1$, we have only a BPS D0-brane with $\Phi^\alpha=0$, which is localized at the origin. However, in the D2-brane picture, the D-brane seem to be localized on the S^2 which does not contain the origin. Moreover, for $n=0$, there are nothing, i.e. D-branes completely disappear by the tachyon condensation. These seem strange in the D2-brane picture, however, we will see these are consistent in section 4.

It is important to note that this fuzzy S^2 brane is off-shell except for $|n|=1$. In order to get the solution of the equations of motions, we have to turn on the RR-flux as in [4].

This might not change the above discussion essentially since the RR-flux considered in [4] only change the Chern-Simons term. If the RR flux is small, R will be proportional to the strength of the RR flux. See also [25]. If we want to consider the on-shell fields, we can consider the D3-D1 bound state where the fuzzy S^2 appear in the D1-brane world volume theory near the D3-branes. In this case it may be the solution of the equations of motions and the problem discussed here still appears, i.e. pictures in the D3-brane and the D1-branes seem different, especially for $|n| = 1$. It is also resolved in the same way discussed in this paper [18].

3.2 Flat noncommutative D-brane

In this subsection, we will consider the flat noncommutative D($2p$)-brane. In [19], it was shown that the boundary state of the flat D($2p$)-brane with the constant field strength F on the world volume is equivalent to the boundary state of the infinitely many D0-branes with $[\Phi^\alpha, \Phi^\beta] = i\left(\frac{1}{F}\right)^{\alpha\beta}$. This can be easily extended to the supersting case [8].

On the other hand, as we reviewed in section 2 we know that the flat D($2p$)-brane with the constant field strength F is also equivalent to the infinitely many unstable D0-branes with the non-trivial tachyon condensation [8]. Combining these two equivalences, we can conclude the infinitely many unstable D0-branes with the non-trivial tachyon condensation are equivalent to the infinitely many BPS D0-branes with $[\Phi^\alpha, \Phi^\beta] = i\left(\frac{1}{F}\right)^{\alpha\beta}$. Now we can directly derive this equivalence by considering the disappearance of the unstable D0-branes by the tachyon condensation. Indeed, as noted before, this was essentially done in [20]. For example in type-IIA string theory, the tachyon of the unstable D0$-\overline{\text{D0}}$ branes is the Dirac operator for the flat space with constant field strength which has a gap between the lowest Landau level and excited states. Actually, the hamiltonian is

$$\widehat{H}_0 = 1_{2\times 2}\left((\widehat{p}_1 - F\widehat{x}_2)^2 + \widehat{p}_2^2\right) - \sigma_3 F_{12}\,, \tag{3.18}$$

and the lowest Landau level for $F_{12} > 0$ is

$$|\,k\,\rangle \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp\left(ikx^1 - \frac{F}{2}\left(x^2 - \frac{k}{F}\right)^2\right) \tag{3.19}$$

in a gauge condition. Therefore in the $u \to \infty$ limit, we have infinitely many BPS D0-branes (or $\overline{\text{D0}}$-branes) which corresponds to the lowest Landau level out of the infinitely many D0$-\overline{\text{D0}}$ pairs. The states in the lowest Landau level are parametrized by p real numbers which correspond to momenta, and, as shown in [20], the coordinates of the remaining D0-branes become

$$(\tilde{\Phi}^1)_{kk'} = -i\frac{\partial}{\partial k}\delta(k-k'), \quad (\tilde{\Phi}^2)_{kk'} = \frac{k}{F}\delta(k-k'), \tag{3.20}$$

where we considered $p = 1$ case for simplicity. This correctly reproduce the commutation relation

$$\left[\tilde{\Phi}^1, \tilde{\Phi}^2\right] = i\frac{1}{F_{12}}\,. \tag{3.21}$$

Note that it becomes the solution of the equation of motion in the $u \to \infty$ limit since we know the configuration (3.21) is on-shell.

The flat noncommutative D-brane is also obtained from the fuzzy D-brane on S^2 by taking some limit. In fact, taking $n \to \infty$ and $\gamma \equiv \frac{4R^2}{n}$ fixed, we have $[\tilde{\Phi}^1, \tilde{\Phi}^2] = i\gamma$ near $\tilde{\Phi}^3 \sim R$, $\tilde{\Phi}^1 \sim \tilde{\Phi}^2 \sim 0$.

4. Noncommutativity and tachyon of D0-branes

In this section, we will see the tachyon condensation of the unstable D0-branes induces the fuzziness to the world volume of the Dp-brane where the parameter u controls the fuzziness. We will also see that the Dp-brane is commutative and localized for $u \to 0$ and becomes noncommutative for $u > 0$.

Let us consider the S^2 case [24, 4], which we have studied in the the subsection 3.1, for illustrative purposes since generalizations to other cases are straightforward.

At $u = 0$, there is no tachyon condensation on the unstable D0-branes. Therefore the boundary state is a simple sum of the boundary states of the D0-branes which have definite positions and each D0-brane corresponds to an $\widehat{x}^\mu$ eigenstate.[9] These D0-branes are commutative and are localized on the sphere $\sum_{\mu=1}^{3}(x^\mu)^2 = R^2$.

Once we turn on nonzero u, the exponent in the boundary interaction contains the operators which are noncommutative each other, namely the tachyon does not commute with $\Phi^\alpha = \widehat{x}^\alpha$. This noncommutativity between the tachyon and the position operators leads the noncommutativity or the fuzziness of the Dp-brane world volume. Actually, replacing $P^\mu(\sigma)$ to its zero mode $\hat{p}_0^\mu$, we find that

$$\left(\sum_{\mu=1}^{3}(\hat{x}_0^\mu)^2 - R^2\right) e^{-S_b}|_{P(\sigma)=\hat{p}_0} \,|\, x=0\,\rangle = \left(\sum_{\mu=1}^{3}(\hat{x}_0^\mu)^2 - R^2\right) \mathrm{Tr}\left(e^{-i\Phi^\mu \hat{p}_0^\mu - 2\pi u^2 D^2}\right) |\, x=0\,\rangle$$
$$\neq 0\,. \tag{4.1}$$

For $u \gg 1$, the eigen states of D are more appropriate basis than the eigen states of $\widehat{x}$. For $u = \infty$, we have n BPS D0-branes which have non-commutative coordinates $\tilde{\Phi}^\mu$ (for $n > 1$) as we have seen in section 3. We can compute the charge density, the stress-energy tensor and other sources for the closed string modes which are expressed as $\langle\, 0 \,|\, V e^{-S_b}\, |x\!=\!0\rangle$ in the boundary state formalism where V is the polynomials of the creation operators corresponding to the closed string modes [26]. Indeed, it was shown in [27] that the D0-brane charge density of the n BPS D0-branes is not localized on S^2 of radius R for finite n using the formula of the matrix model [29]. (See also [30].) Note that we can not have the D0 charge density localized on S^2 of radius R even if we include the possible corrections to the charge density discussed in [27]. This is obvious for $n = 1$ case because all the corrections are commutators and vanish for $n = 1$.

[9] At $u = 0$, the energy density is $CN/(g_s V)$, where C is a numerical factor, N is the number of the unstable D0-branes and V is the volume of the Dp-brane constructed from D0-branes. Thus it is infinite at $u = 0$. In order to avoid this difficulty, we can take u very small, but finite or approximate the Hilbert space as a finite dimensional space. Actually, we can take the Hilbert space spanned by the energy eigen states such that their energies are lower than some fixed energy.

In the path-integral formalism, we use c-numbers instead of the operators. This seems to imply that the Dp-brane is commutative and localized on the S^2 irrespective of u. However, this is not true for $u > 0$. The reason is as follows. For the S^2 case, the path-integral variables are $\theta(\sigma)$ and $\phi(\sigma)$.[10] The coupling $ix^\alpha(\theta(\sigma),\phi(\sigma))P_\alpha(\sigma)$ in the lagrangian $L(x,\dot{x})$ gives a factor

$$\exp\left(-i\left(\int d\sigma x^\alpha(\theta(\sigma),\phi(\sigma))\right)\hat{p}_0^\alpha\right), \tag{4.2}$$

to the boundary state. Here we are concentrating on the zero mode $\hat{p}_0$ of $P(\sigma)$ and dropping the non-zero modes in order to see the space-time distribution of the Dp-brane.

In the $u \to 0$ limit we can ignore the non-zero modes and Dp-brane is localized. This is because the u-dependent term $-\frac{1}{2u^2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$ in the lagrangian becomes infinite in the limit and then the non-zero modes of $\theta(\sigma)$ and $\phi(\sigma)$ becomes infinitely "massive" and are decoupled. Then we will have $\exp\left(i\left(\int d\sigma x^\alpha(\theta(\sigma),\phi(\sigma))\right)P^\alpha(\sigma)\right) = \exp\left(ix^\alpha(\theta_0,\phi_0)\hat{p}_0^\alpha\right)$, where θ_0, ϕ_0 are the zero modes of $\theta(\sigma), \phi(\sigma)$, therefore, the D$p$-brane is localized on the S^2. This is consistent with the result in the D0-brane picture.

However, for $u > 0$ the non-zero modes of $\theta(\sigma)$ and $\phi(\sigma)$ survive in $\int d\sigma x^\alpha(\theta(\sigma),\phi(\sigma))$ because x^α is not linear in $\theta(\sigma)$ and $\phi(\sigma)$. Thus the integrations of the non-zero modes of $\theta(\sigma)$ and $\phi(\sigma)$ make the $\hat{p}_0$ dependent factor (4.2) very complicated, which may give the noncommutativity or fuzziness of the Dp-brane world volume! Note that the non-zero modes of $P(\sigma)$ are coupled to $x(\sigma)$ and then the space-time distribution of the Dp-brane depends on what we probe it with. (For example, we will have different space-time distributions using the couplings to dilaton, graviton, RR fields, and so on.)

This consideration can be clearly applied to any curved Dp-brane. On the other hand, for the flat Dp-brane, $x^\alpha(\sigma)$ itself is the path-integration variable and $\int d\sigma x^\alpha(\sigma) = x_0^\alpha$. Therefore the boundary state has the factor $\exp\left(ix_0^\alpha\hat{p}_0^\alpha\right)$ which mean Dp-brane is localized on the hyperplane for any u as expected. Nevertheless, we can still think it is noncommutative for $u > 0$. Actually, in the D0-brane picture, the constituents D0-branes becomes noncommutative when $u > 0$ even for the flat case. The point is that the flat Dp-brane looks commutative since the noncommutativity is only within the hyperplane. In the BPS limit $u \to \infty$, the noncommutative parameter becomes $1/F$ when we turn the flux on the Dp-brane world volume. This noncommutativity becomes infinite in the $F \to 0$ limit, therefore the flat Dp-brane can be considered to have a maximal noncommutativity in the D0-brane picture. Note that we can think the world volume of the Dp-brane with the flux is either commutative or noncommutative because of the Seiberg-Witten map. The picture here is consistent with the latter one.

Let us study the boundary conditions which the boundary state satisfies. Using $X^\mu(\sigma)\,|\,x=0\,\rangle = 0$ and the $X^\mu(\sigma) = i\frac{\partial}{\partial P^\mu(\sigma)}$, we find

$$\lim_{\epsilon\to 0}\left(\left(\sum_{\mu=1}^{3}X^\mu(\sigma+\epsilon)X^\mu(\sigma) - R^2\right)e^{-S_b}\,|\,x=0\,\rangle\right) = 0\,, \tag{4.3}$$

[10] Here we ignore the fermionic variables since they are irrelevant for the discussion in this section.

for any finite u, which means that the boundary state satisfies the Dirichlet condition on the S^2 with the radius R. In the path-integral formalism (4.3) is obvious. In the operator formalism, we also see that (4.3) is satisfied because of the path-ordering in the trace.

On the other hand, in the $u \to \infty$ limit we have (3.17) which means that the effective radius of S^2 which D2-brane wraps is less than R. This seems puzzling, however, (4.3) does not mean $\left(\sum_{\mu=1}^{3}(X_0^\mu)^2 - R^2\right) e^{-S_b}\,|\,x=0\,\rangle = 0$, where $X_0^\mu = \frac{1}{2\pi}\int d\sigma X^\mu(\sigma)$, except for $u \to 0$, so the D2-brane does not located on the S^2 with the radius R in spite of (4.3). Actually, the average $\bar{x}^\mu = \frac{1}{2\pi}\int d\sigma x^\mu(\sigma)$ of $x^\mu(\sigma)$ satisfying $\sum_{\mu=1}^{3}(x^\mu(\sigma))^2 = R^2$ always satisfies $\sum_{\mu=1}^{3}(\bar{x}^\mu)^2 < R^2$ except it is a constant map, so we can intuitively understand why the effective radius of S^2 is less than R.

There is another problem for $|n| = 1$. In this case, we have $X^\mu(\sigma)e^{-S_b}\,|\,x=0\,\rangle = 0$, $(\mu = 1, \ldots, 3)$ in the $u \to \infty$ limit since the boundary state is a BPS D0-brane at the origin. This seems to contradict (4.3). However, (4.3) is not valid in the $u \to \infty$ limit, namely, the $\epsilon \to 0$ limit and the $u \to \infty$ limit do not commute each other. Therefore there is no contradiction. In order to see the two limits do not commute each other explicitly, let us consider the two unstable D0-branes system (3.5). The boundary state for this satisfies

$$\lim_{\epsilon\to 0}(X(\sigma+\epsilon)X(\sigma) - 1)e^{-S_b}\,|\,x=0\,\rangle = 0 \tag{4.4}$$

for finite u and $X(\sigma)\,e^{-S_b}\,|\,x=0\,\rangle = 0$ for $u \to \infty$. At the lowest order in $P(\sigma)$, we have

$$X(\sigma+\epsilon)X(\sigma)e^{-S_b}\,|\,x=0\,\rangle = \mathrm{Tr}\left(e^{-u^2(2\pi-\epsilon)\mathrm{diag}(2,0)}\sigma_1 e^{-\epsilon u^2 \mathrm{diag}(2,0)}\sigma_1\right)|\,x=0\,\rangle + \mathcal{O}(P(\sigma))\,. \tag{4.5}$$

Taking $u \to \infty$ with keeping $\beta \equiv \epsilon u^2$ finite, we have

$$X(\sigma+\epsilon)X(\sigma)e^{-S_b}\,|\,x=0\,\rangle = e^{-2\beta}\,|\,x=0\,\rangle + \mathcal{O}(P(\sigma))\,. \tag{4.6}$$

Thus for very large, but finite u, $\lim_{\epsilon\to 0}(X(\sigma+\epsilon)X(\sigma) - 1)e^{-S_b}\,|\,x=0\,\rangle \sim 0 + \mathcal{O}(P(\sigma))$. However, if we take $u \to \infty$ first, which means $\beta \to \infty$, $\lim_{\epsilon\to 0}(X(\sigma+\epsilon)X(\sigma))e^{-S_b}\,|\,x=0\,\rangle = 0 + \mathcal{O}(P(\sigma))$ which is consistent with $X(\sigma)e^{-S_b}\,|\,x=0\,\rangle = 0$. From this observation, we expect for $\epsilon \ll 1/u^2 \ll 1$ we have $(X(\sigma+\epsilon)X(\sigma) - 1)\,e^{-S_b}\,|\,x=0\,\rangle \sim 0$ and for $1 \gg \epsilon \gg 1/u^2$ we have $X(\sigma+\epsilon)X(\sigma)\,e^{-S_b}\,|x=0\rangle \sim 0$. Note that the $X(\sigma)$ is not a function of σ, but an operator valued function of σ.

In summary, we have a (almost) commutative Dp-brane for $u \ll 1$. As u goes larger the fuzziness of the world volume becomes stronger and the D0-brane picture is more natural for $u \gg 1$. An interesting conclusion is that a curved Dp-brane which has Dirichlet boundary condition on a curved submanifold is not localized on it. This fact resolves the problem for the D2-brane on S^2 with the unit flux.

Some comments are as follows. Taking $F \to \infty$ limit for the flat noncommutative D-brane or $n \to \infty$ limit with R fixed for the fuzzy S^2 D-brane, we have a similar situation as in $u \to 0$ limit, i.e. the localized D-brane. Actually, using the Dp-brane picture, we can see that these have infinite energy because of the infinite field strength, which means that the BPS D0-branes are dominated and other D0-branes which disappear by the tachyon condensation are not important. In fact, the energy gap between the ground states and

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excited state is roughly proportional to F or n. Thus, for finite u, the value of u is not important for $F \to \infty$ and these are like $u \to 0$.

We can see that the energy gap between the ground states and excited states is order $\frac{u^2}{R^2}$ where R is a characteristic length scale of the submanifold which Dp-brane is supposed to wrap. Here we have chosen the vielbein as the induced metric and defined the u as $T = uD$. From this, we expect the parameter $\frac{u}{R}$ controls the fuzziness and should be small if we want to have the localized brane. Moreover, in order to have the Dp-brane we should take $1 \ll u$, which can be seen from the boundary state in the path-integral formalism. Thus, we expect that we have the Dp-brane wrapping the manifold which locally looks like the flat Dp-brane if we can take

$$1 \ll u \ll R\,, \tag{4.7}$$

which implies $1 \ll R$ or $\sqrt{\alpha'} \ll R$ if we recover the α'. Note that we have defined T and u dimensionless. We can not have such a Dp-brane if $R \sim \sqrt{\alpha'}$ because of the noncommutativity. Let us consider a D2-brane on S^2 of radius $R \sim \sqrt{\alpha'}$ with unit flux as an example. If we want to get the D2-brane localized on S^2, it should have finite u since it shrinks to the origin for $u = \infty$. Thus we can not have the D2-brane localized on S^2 which is locally like the flat D2-brane. One might think the D2-brane for $u \to \infty$ is still described by the DBI action on S^2 of radius R. However, the DBI action is not reliable, for example, for the D2-brane with finite n on S^2 in the limit $u \to \infty$ since the (higher derivative) corrections to the DBI action can not be neglected. In particular, there will be $\mathcal{O}(\frac{u}{R})$ terms in the action. Indeed, it should be described by the action of the n BPS D0-branes which are effectively on S^2 of radius R_{eff}. Especially for $|n| = 1$ and $n = 0$, the picture from the DBI action is completely wrong.

Here it should be emphasized that this conclusion does not mean the usual analysis by the DBI action is wrong. Indeed, it is well known that the DBI action is reliable only if higher derivative terms can be neglected, i.e. $R/\sqrt{\alpha'} \gg 1$. In this case, we can always take $u \ll R$ and the picture from the DBI action is indeed reliable. As an example, let us consider the Myers effect [4], where the RR flux is turned on. The analysis is reliable for $|n| \gg 1$, i.e. large magnetic flux. This means large R by the on-shell condition although we do not require on-shell condition in general. On the other hand, it is also known that the DBI analysis is wrong if $|n|$ is finite.

Finally, we can think that the parameter $1/u$ represents the uncertainty of $\widehat{p}$ [9]. Thus the position is definite for $u \to 0$, on the other hand, the position is ambiguous for $u \to \infty$. This agrees with its interpretation as the parameter controlling the noncommutativity of the world volume.

4.1 Diffeomorphism and Seiberg-Witten map

In this subsection, we will discuss the diffeomorphism of the world volume of the Dp-brane. In the unstable D0-brane picture, it is realized as a subgroup of the U(N) gauge symmetry of the N D0-branes [6]. This U(N) gauge symmetry is unbroken if T, Φ^μ and other fields are proportional to the $N \times N$ identity matrix. For the Dp-brane configuration, The U(N) gauge symmetry is broken to the over all U(1), but a subgroup becomes gauge symmetry of the Dp-brane, which contains the U(1) gauge symmetry $e^{i\lambda(\widehat{x})}$, (the local Lorentz symmetry $e^{[\gamma^A,\gamma^B]c_{AB}}$) and the diffeomorphism $e^{i\eta^\alpha(\widehat{x})\widehat{p}_\alpha}$ [6, 8].[11]

If we include u into the definition of the vielbein $e^\mu_A \to ue^\mu_A$, $u \ll 1$ corresponds to the large world volume metric $g_{\mu\nu} = e^A_\mu e^B_\nu \delta_{AB} \gg 1$. As we have seen, $u \ll 1$ is a geometric region, i.e. the world volume is commutative and localized. The large metric expansion of the BSFT action $S_{BSFT} = \frac{2\pi}{g_s} \langle\, 0 \,|\, e^{-S_b} \,|\, x = 0 \,\rangle$ is [6]

$$S_{BSFT} \sim \frac{1}{g_s} \int dt \int dx^p \sqrt{g} \left(1 + 2\log 2 G_{\mu\nu} \frac{\partial f^\mu(x)}{\partial x^\alpha} \frac{\partial f^\nu(x)}{\partial x^\beta} g^{\alpha\beta} + \mathcal{O}(g^{-2}) \right) \tag{4.8}$$

where $G_{\mu\nu} = \delta_{\mu\nu}$ and $f^\mu(x)$ is the embedding of the world volume to the flat space. This action is diffeomorphism invariant. From the first term, we see that the metric is related to the density of the unstable D-branes in this region of u since the unstable D0-branes are almost independent each other and the energy of the N D0-branes is proportional to N.

On the other hand, for $u \gg 1$, we should have the world volume action on the Dp-branes, for example, the DBI action for the flat world volume with the constant F. Actually, we know the equivalence between the boundary states. It means that the S_{BSFT} becomes the world volume action on the Dp-branes in the $u \to \infty$ limit because the disk partition function, $\frac{2\pi}{g_s}\langle 0 \,|\, Dp \,\rangle$, gives the the effective action of the Dp-brane [32]. In leading order in $1/R$, we will have the DBI action or the Nambu-Goto action neglecting the gauge fields and other fields,

$$S_{BSFT} \sim \frac{2\pi}{g_s} \int dt \int dx^p \sqrt{\det \left(G_{\mu\nu} \frac{\partial f^\mu(x)}{\partial x^\alpha} \frac{\partial f^\nu(x)}{\partial x^\beta} \right)}. \tag{4.9}$$

We note that the volume factor in the (4.9) is much smaller than $\sqrt{g}$ in (4.8) since the induced metric is finite, but g_{ij} is very large for $u \ll 1$. We also note that (4.9) is valid only if (4.7) is satisfied. This is because there will be $\mathcal{O}(\frac{u}{R})$ terms and these terms can not be neglected if (4.7) is not satisfied.

Now let us consider a hamiltonian $\widehat{H}_0$ with a gap and take the $u \to \infty$ limit. Then the Hilbert space is reduced to the space spanned by the zero modes. We denote the projection operator to this space as P, namely $P = \sum_{a=1}^{N_0} |\, a \,\rangle \langle\, a \,|$. Then by the gauge transformation $U \in \mathrm{U}(N)$, which includes the diffeomorphism, the projection operator becomes $P' = UPU^\dagger$. Thus it gives another description of the system. However, in $[P, U] = 0$ case, an operator $U' = PUP$ in the reduced space is unitary and we can interpret this as a generator of the residual gauge symmetry.

Let us concentrate on the flat noncommutative D($2p$)-brane. In this case, there are a commutative and a noncommutative descriptions which are related by the the Seiberg-Witten map [3]. Furthermore, the SW map can be almost understood as a diffeomorphism [34, 33, 35]. Since the diffeomorphism is realized as the U(N) gauge symmetry, i.e. unitary transformation of the basis, in our picture, we expect that the Seiberg-Witten map

[11] This realization of the diffeomorphism is similar to the spectral action principle [31]. If we formally consider $\langle\, Dp \,|\, e^{-S_b} \,|\, x = 0 \,\rangle \sim \mathrm{Tr}(e^{-u^2 \widehat{H}_0})$ and take $u \ll 1$, we have the spectral action without fermion although the D-brane action is different from it.

is almost realized as the U(N) gauge symmetry in our picture also. Consider the D2-brane constructed from the unstable D0-branes with

$$T = u\gamma^\alpha \left(\widehat{p}_\alpha + \frac{1}{2} f_{\alpha\beta} \widehat{x}^\beta + A_\alpha(\widehat{x}) \right), \qquad \Phi^\alpha = \widehat{x}^\alpha \,, \tag{4.10}$$

where $f_{\alpha\beta} = -f_{\beta\alpha}$. Then, $f_{\alpha\beta}$ is background flux and $A_\alpha(x)$ is the fluctuation of the gauge field around it. The diffeomorphism discussed in [34, 33, 35] is given by $U = e^{i\eta^\alpha(\widehat{x})\widehat{p}_\alpha}$ such that $U^\dagger T U = u\gamma^A e_A^\alpha(\widehat{x})(\widehat{p}_\alpha + \frac{i}{8}\omega_{BC,\alpha}[\gamma^B, \gamma^C] + \frac{1}{2} f_{\alpha\beta}\widehat{x}^\beta)$ and we define $\hat{A}_\alpha(x)$ as $U^\dagger \Phi^\alpha U = \widehat{x}^\alpha + (f^{-1})^{\alpha\beta} \hat{A}_\beta(\widehat{x})$. Note that there should be a map between $A_\alpha(x)$ and $\hat{A}_\alpha(x)$ up to the gauge transformation. The vielbein $e_A^\alpha(\widehat{x})$ will be almost reduced to trivial one in the $u \to \infty$ limit, the D2-brane is almost described by

$$T = u\gamma^\alpha \left(\widehat{p}_\alpha + \frac{1}{2} f_{\alpha\beta} \widehat{x}^\beta \right), \qquad \Phi^\alpha = \widehat{x}^\alpha + \left(f^{-1}\right)^{\alpha\beta} \hat{A}_\beta(\widehat{x}) \tag{4.11}$$

in the $u \to \infty$ limit. Here the boundary states for these two configurations are actually those considered in [35]. Thus the map between $A_\alpha(x)$ and $\hat{A}_\alpha(x)$ is almost the SW map as shown in [35]. In order to consider the obtain the exact SW map, we need to have the zero modes of T in (4.10) and evaluate $(\tilde{\Phi}^\alpha)_{ab} = \langle\, a \,|\, \widehat{x}^\alpha \,|\, b \,\rangle$. Then the $\hat{A}_\alpha(x)$ should be given by the formula $\tilde{\Phi}^\alpha = \widehat{x}^\alpha + (f^{-1})^{\alpha\beta} \hat{A}_\beta(\widehat{x})$ where $\widehat{x}$ is the operator on the reduced Hilbert space which is different from $\widehat{x}$ in (4.10). This will give another representation of the SW map. [12]

It is expected that the diffeomorphism realized as a subgroup of U(N) in the N unstable D0-branes will be reduced to the area preserving diffeomorphism realized as a subgroup of U(N_0) in the N_0 BPS D0-branes after taking $u \to \infty$ limit. It is an interesting problem to explicitly show this.

5. Conclusions

In this paper, we have studied the fuzzy or noncommutative Dp-branes in terms of infinitely many unstable D0-branes and showed that the condensation of the tachyon of the unstable D0-branes induces the noncommutativity. We have also showed that a boundary state for a Dp-brane satisfying the Dirichlet boundary condition on a curved submanifold embedded in the flat space is not localized on the submanifold. Although, we have considered a few examples, it should be possible to generalize to the D-brane wrapping S^4 [36], supertubes [37] and others [38].

The $\frac{1}{u^2}(\dot{x^\alpha})^2$ term (2.14) on the boundary of the world sheet can be regarded as the trace part of the massive symmetric tensor fields, $W_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta$. It was argued that this trace part of the massive tensor can not be turned on because it gives the nonzero one point-function which corresponds to a linear term in the effective action [28]. However, from the D0-brane point of view, this term corresponds to the tachyon and there is no apparent problem with this term, especially, the effective action does not have a term

[12] The SW map is not unique [43] and indeed, the SW maps for the bosonic case and the superstring case are different [44, 45]. This difference might come from the subtlety of the UV regularization.

linear in the tachyon. This is valid at least if we regularize the Hilbert space by a finite dimensional space. The term is proportional to $1/u^2$ which means that (2.14) is expanded by the inverse of the tachyon of the D0-branes. Actually, the net effect of the problematic one-point function in this case is a change of the normalization of the boundary state by a factor $\exp(\frac{\pi}{4u^2})$ which can not be expanded around $u = 0$. Hence it might be possible to think (2.14) as a background. The role of this term may be interesting to study.

As mentioned in the introduction, the noncommutative geometry of [5] is represented by the spectral triple which contains the Dirac operator or an analogue of it. Because of it, we can construct the analogue of the riemannian geometry. The Dirac operator corresponds to the tachyon operator in our unstable D0-branes [6]. However, the noncommutative plane and the fuzzy sphere in the string theory [1, 3, 4] do not have such an object naturally. Indeed, as we have seen, the information of the metric or the Dirac operator will be almost dropped in the $u \to \infty$ limit, which leads the noncommutative plane and the fuzzy sphere. However, even in the $u \to \infty$ limit, the Dirac operator plays important role, for example, in the computation of the RR-charges which is directly related to the Atiyah-Singer index theorem [8]. Moreover, if we can incorporate the space-time fermions, the Dirac operator may play important role as in [31]. From the point of view of the Connes' noncommutative geometry, it may be interesting to study how the commutative geometry with the nontrivial gauge field is related to the noncommutative geometry. We hope to come back to this in the future.

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A. Boundary state

In this section we review the boundary states for the D-branes which are realized in the closed string Hilbert space.[13]

The closed string operators at boundary of the world sheet are,

$$X^\mu(\sigma) = \widehat{x}_0^\mu + \sum_{m\neq 0} \frac{1}{\sqrt{|m|}} \left(a_m^\mu e^{-im\sigma} + \widetilde{a}_m^\mu e^{im\sigma} \right), \tag{A.1}$$

$$P^\mu(\sigma) = \widehat{p}_0^\mu - i \sum_{m\neq 0} \sqrt{|m|} \left(a_m^\mu e^{-im\sigma} - \widetilde{a}_m^\mu e^{im\sigma} \right), \tag{A.2}$$

where

$$a_{-m}^\mu = a_m^{\mu\dagger}, \qquad \widetilde{a}_{-m}^\mu = \widetilde{a}_m^{\mu\dagger}, \qquad (m > 0), \tag{A.3}$$

[13]We omit the ghost part which does not play any role in this paper because we do not consider the loop amplitudes. We also omit the factor which depends on the time-direction, which is universal for time-independent configuration. See [39] for those.

which acts on the closed string Hilbert space. The commutation relations are

$$\left[X^\mu(\sigma), P^\nu(\sigma')\right] = i\delta^{\mu\nu}\delta(\sigma-\sigma')\,, \tag{A.4}$$

$$\left[a^\mu_m, a^{\nu\dagger}_n\right] = \left[\widetilde{a}^\mu_m, \widetilde{a}^{\nu\dagger}_n\right] = \delta_{m,n}\delta^{\mu\nu}\,. \tag{A.5}$$

The closed string Fock vacuum $|\,0\,\rangle$ is defined such that

$$\widehat{p}^\mu_0\,|\,0\,\rangle = a^\mu_m\,|\,0\,\rangle = \widetilde{a}^\mu_m\,|\,0\,\rangle = 0 \qquad (m>0)\,. \tag{A.6}$$

The coherent state of $X^\mu(\sigma)$ is defined as

$$X^\mu(\sigma)\,|\,x\,\rangle = x^\mu(\sigma)\,|\,x\,\rangle\,, \tag{A.7}$$

where

$$x^\mu(\sigma) = x^\mu_0 + \sum_{m\neq 0}\frac{1}{\sqrt{|m|}}x^\mu_m e^{-im\sigma}\,, \tag{A.8}$$

which is explicitly written as

$$|\,x\,\rangle = \exp\left(\sum_{m=1}^{\infty}\left(-\frac{1}{2}x_{-m}x_m - a^\dagger_m\widetilde{a}^\dagger_m + a^\dagger_m x_m + x_{-m}\widetilde{a}^\dagger_m\right)\right)\Big|x_0\Big\rangle, \tag{A.9}$$

where $|\,x_0\,\rangle$ satisfies $\widehat{x}^\mu_0\,|\,x_0\,\rangle = x^\mu_0\,|\,x_0\,\rangle$ and $a^\mu_m\,|\,x_0\,\rangle = \widetilde{a}^\mu_m\,|\,x_0\,\rangle = 0$. Then the boundary state for a D0-brane at the origin is given as $|\,D0\,\rangle \equiv |\,x=0\,\rangle = |\,x\,\rangle\,|_{x_0=x_m=0}$ which satisfies

$$X^\mu(\sigma)\,|\,x=0\,\rangle = 0\,. \tag{A.10}$$

It, of course, implies the Dirichlet boundary condition $\dot{X}^\mu(\sigma)\,|\,x=0\,\rangle = 0$.

The boundary states for Dp-branes is defined by

$$|\,Dp\,\rangle \equiv \int [dx^\alpha]\,|\,x^\alpha, x^i=0\,\rangle \tag{A.11}$$

where the superscript $\alpha = 1,\ldots,p$ and $i = p+1,\ldots,25$ represent the direction tangent and transverse to the D-brane, respectively. In fact, we can check that this satisfies the Neumann boundary condition, $P_\alpha(\sigma)\,|\,Dp\,\rangle = 0$.

The fields on the D-brane world-volume can be turned on through a boundary interaction. Then, the boundary state $|\,Dp\,\rangle$ are modified as

$$|\,Dp\,\rangle_{S_b} = e^{-S_b}\,|\,Dp\,\rangle \tag{A.12}$$

where e^{-S_b} is an abbreviation for the boundary interaction:

$$e^{-S_b} = \mathrm{Tr}\,\mathrm{P}\exp\left\{\oint d\sigma\left(-S_b(X^\alpha(\sigma), P^i(\sigma))\right)\right\}. \tag{A.13}$$

We can rewrite it as

$$e^{-S_b}\,|\,Dp\,\rangle = \int [dx^\alpha]\,\mathrm{Tr}\,\mathrm{P}\exp\left\{\oint d\sigma\left(-S_b(x^\alpha(\sigma), P^i(\sigma))\right)\right\}|\,x^\alpha, x^i=0\,\rangle\,. \tag{A.14}$$

Here $S_b(x^\alpha(\sigma), P^i(\sigma))$ is $N\times N$ matrix due to the Chan-Paton index and $\mathrm{Tr\,P}$ is the path-ordered trace for this index, where N is the number of the Dp-branes we consider. Note that if there is no boundary interaction, i.e. $S_b=0$, we have $e^{-S_b}\,|\,Dp\,\rangle = N\,|\,Dp\,\rangle = \sum_{n=1}^{N}|\,Dp\,\rangle$.[14]

The boundary interaction for the gauge fields on the Dp-branes is given as [40]

$$e^{-S_b} = \mathrm{Tr\,P}\exp\left\{\oint d\sigma\left(-A_\alpha(X)\dot{X}^\alpha\right)\right\}. \tag{A.15}$$

Similarly, the boundary interaction for the tachyons T and the massless scalars Φ^i on the D0-branes is given as

$$e^{-S_b} = \mathrm{Tr\,P}\exp\left\{\oint d\sigma\left(-T - i\Phi^i P_i\right)\right\}, \tag{A.16}$$

where T and Φ^i are $N\times N$ matrices and σ independent. In particular, if we set $T=0$ and $\Phi^i = \mathrm{diag}(a_1^i,\ldots,a_N^i)$, we have $e^{-S_b}\,|\,D0\,\rangle = \sum_{n=1}^{N} e^{i\hat{p}_0^i a_n^i}\,|\,D0\,\rangle$, i.e. n-th D0-brane is located at $x^i = a_n^i$.

Next we summarize the superstring case very briefly. See [6] for details. In super string, the boundary states for D-branes are obtained as a linear combinations of the boundary states defined by

$$|\,Bp;\pm\,\rangle = \int [dx^\alpha][d\psi^\alpha]\,|\,x^\alpha, x^i = 0\,\rangle\,|\,\psi^\alpha,\psi^i = 0;\pm\,\rangle\,, \tag{A.17}$$

where the superscript α and i represent the direction tangent and transverse to the D-brane, respectively. The state $|\,\theta\,\rangle$ is the coherent state for the world sheet fermions satisfying

$$\Psi_\pm^\mu(\sigma)\,|\,\psi;\pm\,\rangle = \psi^\mu(\sigma)\,|\,\psi;\pm\,\rangle\,, \tag{A.18}$$

where

$$\Psi_\pm^\mu(\sigma) = \Psi^\mu(\sigma) \pm i\widetilde{\Psi}^\mu(\sigma) = \sum_r (\Psi_r^\mu e^{-ir\sigma} \pm i\widetilde{\Psi}_r^\mu e^{ir\sigma})\,, \qquad \psi^\mu(\sigma) = \sum_r \psi_r^\mu e^{-ir\sigma}. \tag{A.19}$$

In this paper, we will omit the subscript $\pm$ because it does not play any role in this paper. We also introduce $\Pi_\mu(\sigma)$ which is the conjugate momentum of $\Psi^\mu(\sigma)$.

For simplicity, we consider N D0$-\overline{\mathrm{D0}}$ pairs in type-IIA and set $\Phi^\mu = \tilde{\Phi}^\mu$ where $N\times N$ matrices Φ and $\tilde{\Phi}$ are massless scalars on D0 and $\overline{\mathrm{D0}}$, respectively. We define a $2N\times 2N$ matrix $\begin{pmatrix}\Phi^\mu & 0\\ 0 & \tilde{\Phi}^\mu\end{pmatrix} = \Phi^\mu 1_{2\times 2}$ and we will denote it as Φ^μ in this paper. We also denote a $2N\times 2N$ hermitean matrix $\begin{pmatrix}0 & T\\ T^\dagger & 0\end{pmatrix}$ as T for notational simplicity. Then the boundary interaction is [17, 16]

$$e^{-S_b} = \mathrm{Tr\,P}\exp\int d\sigma\left(-i\Phi^\mu P_\mu(\sigma) - \frac{1}{2}[\Phi^\mu,\Phi^\nu]\Pi_\mu(\sigma)\Pi_\nu(\sigma) - T^2 + \Pi_\mu(\sigma)[T,\Phi^\mu]\right), \tag{A.20}$$

[14] If we take $2\pi T = -\log N$, the boundary state of the N Dp-branes becomes $Ne^{-2\pi T}\,|\,Dp\,\rangle = |\,Dp\,\rangle$. However, this is different from one Dp-brane which has different fluctuations on its world volume.

where $\mathrm{Tr}\,\mathrm{P}$ is the path-ordered trace for the $2N \times 2N$ matrix. From this, we also see that boundary interaction for the N_0 BPS D0-branes is

$$e^{-S_b} = \mathrm{Tr}\,\mathrm{P}\exp\int d\sigma\left(-i\Phi^\mu P_\mu(\sigma) - \frac{1}{2}[\Phi^\mu,\Phi^\nu]\Pi_\mu(\sigma)\Pi_\nu(\sigma)\right), \tag{A.21}$$

where $\mathrm{Tr}\,\mathrm{P}$ is the path-ordered trace for the $N_0 \times N_0$ matrix.

B. Normalization of the boundary state

In this appendix, we discuss about the normalization of the boundary state.

First we introduce a normalization factor A to the path-integral measure in (2.5) as $\mathrm{Tr}\,\mathrm{P}\,e^{-i\int d\sigma H} = A\int[dx]e^{i\int d\sigma L}$, where

$$A = \frac{\langle\, 0\,|\,\mathrm{Tr}\,\mathrm{P}\,e^{-i\int d\sigma \widehat{H}}\,|\,D0\,\rangle}{\langle\, 0\,|\int[dx]e^{i\int d\sigma L}\,|\,D0\,\rangle}. \tag{B.1}$$

Formally we can rewire it as $A = \mathrm{Tr}\,e^{-u^2\widehat{H}_0}/\int[dx(\sigma)]e^{-\int d\sigma L_0}$. Then in (2.5) we can use a path-integral measure with any normalization. Especially, we can use the measure for the perturbative theory.

Now let us recall that the BSFT action S_{BSFT} (for on-shell fields) is proportional to the disk partition function [41, 13–17], and the normalization of the BSFT action should be fixed such that it correctly reproduce the tension of the Dp-brane. Using the boundary state, we can define $S_{BSFT} = \frac{2\pi}{g_s}\langle\, 0\,|\,Dp\rangle$ by appropriately choosing the normalization of the path-integral measure in (A.11). For the D(-1)-brane, we have used $\langle\, 0\,|\,D(-1)\rangle = \langle\, 0\,|\,x=0\rangle = \langle\, 0\,|\,e^{-\sum a^\dagger \tilde{a}^\dagger}\,|\,x_0\,\rangle = 1$. Therefore, with this normalization, A should be 1 if (2.5) or its supersting analogue gives the correct tension of the Dp-brane. This condition is equivalent to the condition that the Dp-brane solution in the N D0-branes has correct tension. This was indeed shown in [9] and the correct ration is obtained for superstring. Furthermore, if we use the zeta-function regularization for the path-integral measure [6, 8], we have $A = 1$ though it is rather formal. In this paper, we will use a definition of the path-integral measure $[dx]$ (or its supersymmetric extension) such that $A = 1$.

For the bosonic case, we expect $A = 1$ since the correct tension is obtained in the decent relation [13]. Note that the normalization of the boundary state is related to the constant part of the tachyon $t(x)$ and we should take $t(x)$ such that $T \geq 0$ for bosonic case, i.e. the eigen values of the T are zero or positive. (At least, there should be no eigen value which goes to $-\infty$.) This is because the tachyon potential $V(T)$ goes to $-\infty$ as $T \to -\infty$ and the negative tachyon region may be forbidden although we do not know a correct interpretation of it.

Finally, we note that the disk partition function used in (B.1) has the UV divergence of the world sheet and we should regularize it as usual. This can be done by replacing $P(\sigma)$ by $e^{\epsilon\frac{\partial^2}{\partial\sigma^2}}P(\sigma)$ which is equivalent to the regularization used in [32]. Then the fluctuations $A_\alpha(\widehat{x}), \phi^i(\widehat{x})$ and $t(\widehat{x})$ will depend on the regularization parameter ϵ.

C. Decoupling of the D0-branes

In this appendix we will first consider the validity of the assumption leading (3.2) or (3.3) and then we will consider the Harmonic oscillator case which violates the assumption.

Let us consider the case in bosonic string with $t(x) = 0$. Here $\widehat{H} = u^2 H_0 + i f^\mu(x) P_\mu(\sigma)$. Then we can regard $u^2 H_0$ as the "free hamiltonian" and $H_I = i f^\mu(x) P_\mu(\sigma)$ as the interaction hamiltonian. As usual in the interaction picture, we expand e^{S_b} as

$$e^{-S_b} = \sum_{n=0}^{\infty} \mathrm{Tr} \left(e^{-2\pi u^2 H_0} \prod_{l=1}^{n} \int_0^{2\pi} d\sigma_l \Theta(\sigma_l - \sigma_{l-1}) H_I(\sigma_l) \right) \tag{C.1}$$

where $H_I(\sigma) = i e^{\sigma u^2 H_0} f^\mu(x) e^{-\sigma u^2 H_0} P_\mu(\sigma)$ and $\sigma_0 = 0$. Thus we can evaluate it using the eigen states of H_0 as in [20]. It is easy to see that the most dangerous terms comes from the integrations near $\sigma_l = \sigma_{l-1}$ and they are $\mathcal{O}(\frac{1}{u^2})$. Thus if the summation over the energies is finite, the assumption is valid at least in this perturbation theory. Actually, for the circle or sphere Dp-brane case we can easily see that only a finite number of the energy eigen states contributes because of the rotation symmetry. By adding some smooth terms to $f^\mu(x)$, we can consider the Dp-brane of generic shape and the summation over the energies is finite. Thus we expect that the assumption is valid for the Dp-brane with $t(x) = 0$.

Let us consider the hamiltonian of the Harmonic oscillator in the bosonic string. We take

$$u^2 t = \frac{\alpha^2}{2} \widehat{x}^2 - \frac{1}{2} u\alpha \,. \tag{C.2}$$

Then

$$u^2 \widehat{H}_0 = \frac{u^2}{2} \widehat{p}^2 + \frac{\alpha^2}{2} \widehat{x}^2 - \frac{1}{2} u\alpha \tag{C.3}$$

is a harmonic oscillator with a mass $\sqrt{m}\omega = \alpha$ and a frequency $\frac{1}{\sqrt{m}} = u$. The eigen state $|\, n \,\rangle$ ($n \in Z \geq 0$) has the energy $E_n = u\alpha n$. The expectation value of the position operator for the ground state vanishes, $\langle\, 0 \,|\, \widehat{x} \,|\, 0 \,\rangle = 0$. in the $u \to \infty$ limit one might think only the ground state contributes in the $\mathrm{Tr}\,\mathrm{P}$ as argued in section 3 and obtain

$$|\, B \,\rangle = \mathrm{Tr}\,\mathrm{P}\, e^{-i \int d\sigma \hat{H}(\hat{x},\hat{p})} \,|\, x = 0 \,\rangle = |\, x = 0 \,\rangle \tag{C.4}$$

in the $u \to \infty$ limit. However, this observation is not correct because the excited states do contribute in the $\mathrm{Tr}\,\mathrm{P}$ even in the $u \to \infty$ limit. Indeed, at the second order perturbation theory, $\mathrm{Tr}\,\mathrm{P}\, e^{-i \int d\sigma \hat{H}(\hat{x},\hat{p})}$ is

$$\begin{aligned}
&\sum_{n=0}^{\infty} \int_0^{2\pi} d\sigma_1 \times \\
&\quad \times \int_0^{2\pi} d\sigma_2 \Theta(\sigma_2 - \sigma_1) \,\langle\, n \,|\, e^{-(2\pi - \sigma_2)T} (i\widehat{x} P(\sigma_2)) e^{-(\sigma_2 - \sigma_1)T} (i\widehat{x} P(\sigma_1)) e^{-\sigma_1 T} \,|\, n \,\rangle = \\
&\qquad = \sum_{n,m=0}^{\infty} \int_0^{2\pi} d\sigma_2 \int_0^{\sigma_2} d\sigma_1 e^{-2\pi n u\alpha} e^{(\sigma_2 - \sigma_1)(n-m)u\alpha} \times \\
&\qquad \times \sum_{n=0}^{\infty} \langle\, n \,|\, (i\widehat{x}) \,|\, m \,\rangle \,\langle\, m \,|\, (i\widehat{x}) \,|\, n \,\rangle\, P(\sigma_1) P(\sigma_2) \,,
\end{aligned}$$

and if we set $P(\sigma)$ to its zero mode p_0 for simplicity[15] and take the $u \to \infty$ limit which implies $n = 0$, it is reduced to

$$\sum_m^\infty \int_0^{2\pi} d\sigma_2 \frac{1}{u\alpha m}(1 - e^{-\sigma_2 m u\alpha}) \langle\, 0 \,|\, i\widehat{x} \,|\, m \,\rangle \langle\, m \,|\, i\widehat{x} \,|\, 0 \,\rangle\, p_0^2 =$$
$$= -\frac{2\pi}{u\alpha} |\langle\, 0 \,|\, \widehat{x} \,|\, 1 \,\rangle|^2 p_0^2 + \left(\text{higher order in } \frac{1}{u}\right). \qquad \text{(C.5)}$$

Here, the integrations near $\sigma_1 = \sigma_2$ gave the leading term. We can compute $|\langle\, 0 \,|\, \widehat{x} \,|\, 1 \,\rangle|^2 = \frac{u}{\alpha}$ using a following transformation

$$\widehat{p} = \sqrt{\frac{\alpha}{u}}\widehat{p}', \qquad \widehat{x} = \sqrt{\frac{u}{\alpha}}\widehat{x}', \qquad \text{(C.6)}$$

which keeps the commutation relation invariant $[\widehat{x}', \widehat{p}'] = i$ and $T = u\alpha(\frac{\widehat{p}'^2}{2} + \frac{\widehat{x}'^2}{2} - \frac{1}{2})$. Therefore, it becomes

$$-\frac{2\pi}{u\alpha} |\langle\, 0 \,|\, \widehat{x} \,|\, 1 \,\rangle|^2 p_0^2 = -2\pi \frac{p_0^2}{\alpha^2}, \qquad \text{(C.7)}$$

which is indeed finite in the $u \to \infty$ limit. This means that the infinitely many unstable D0-branes are not reduced to a D0-brane represented by the ground state $|\, 0 \,\rangle$ and the system is not localized at the point in the $u \to \infty$ limit.

Actually, the boundary interaction is written as

$$\mathrm{Tr\,P}\, e^{-\int d\sigma \left(u\alpha \left(\frac{\widehat{p}'^2}{2} + \frac{\widehat{x}'^2}{2} - 1 \right) - i\widehat{x}' \sqrt{u}\alpha P(\sigma) \right)},$$
$$= \mathrm{Tr\,P}\, e^{-\int d\sigma \left(u\alpha \left(\frac{\widehat{p}'^2}{2} + \frac{(\widehat{x}' - i\sqrt{\frac{1}{u\alpha^3}}P(\sigma))^2}{2} - 1 \right) - \frac{1}{2\alpha^2}P(\sigma)^2 \right)}. \qquad \text{(C.8)}$$

If we take $P(\sigma)$ to its zero mode p_0, it is reduced to

$$e^{-\frac{1}{2\alpha^2}p_0^2}, \qquad \text{(C.9)}$$

in the $u \to \infty$ limit and its Fourier transform is $\alpha e^{-\frac{\alpha^2}{2}x_0^2}$. Thus the result is an object smeared in the region $|x| < \frac{1}{\alpha}$.

In the path-integral formalism, we have

$$-L_0 = -\frac{1}{2u^2}\dot{x}(\sigma)^2 - \frac{\alpha^2}{2}x(\sigma)^2 + \frac{1}{2}u\alpha \to_{u\to\infty} -\frac{\alpha^2}{2}x(\sigma)^2 + \frac{1}{2}u\alpha, \qquad \text{(C.10)}$$

which is the tachyon profile studied in the bosonic BSFT [13] and was interpreted as the tachyon kink solution in the D1-brane, which represents a D0-brane in the limit $\alpha \to \infty$. We note that the zero mode sector of it is exactly coincide with the (C.9).

We note that instead of D0-branes in the bosonic string theory we can repeat the above consideration for the non BPS D0-branes in type-IIB string theory. In this case, there is no constant shift like $u\alpha/2$.

[15]This is formally archived by considering $\langle\, D1 \,|\, e^{-ip_0\widehat{x}_0} e^{-S_b} \,|\, x = 0 \,\rangle$.

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